International Conference Mathematical and Computational Biology 2011
International Journal of Modern Physics: Conference Series
Vol. 9 (2012) 574–580

DOI: 10.1142/S2010194512005685

SIMULATION OF KORTEWEG DE VRIES EQUATION

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Korteweg de Vries (KdV) equation has been used as a mathematical model of shallow water waves. In this paper, we present one-, two-, and three-soliton solution of KdV equation. By definition, soliton is a nonlinear wave that maintains its properties (shape and velocity) upon interaction with each other. In order to investigate the behavior of soliton solutions of KdV equation and the interaction process of the two- and three-solitons, computer programs have been successfully simulated. Results from these simulations confirm that the solutions of KdV equation obtained are the soliton solutions.

Keywords: KdV equation; solution of KdV equation; soliton interaction.

1. Introduction

Soliton which is a nonlinear wave is considered as a solitary wave solution of partial differential equation (PDE). The specialty of soliton is that this wave re-emerges and retains its identities with same speed and shape after a fully nonlinear interaction. This dynamical behavior is investigated in this work.

In 1895, D. J. Korteweg and G. de Vries formulated a mathematical model to explain Russell's observation[1]. Both of them derived a new nonlinear equation given by equation (1) where this equation describes the propagation of wave on the surface of a shallow channel[1]. The original Korteweg de Vries (KdV) equation is written as

$$\frac{\partial \eta}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial \xi}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \xi^3}\right). \tag{1}$$

where $\sigma = \frac{1}{3}h^3 - \frac{Th}{\rho g}$, η is the surface elevation of the wave above the equilibrium level, h is equilibrium level, α is a small arbitrary constant related to the uniform motion of the liquid, g is a gravity constant, T is a surface tension of the liquid and ρ is density of the liquid. According to Ablowitz and Clarkson (1991) by making these transformations $t = \frac{1}{2}\sqrt{\frac{g}{h\alpha}}\tau$, $x = -\sigma^{1/2}\xi$ and $u = \frac{1}{2}\eta + \frac{1}{3}\alpha$, equation (1) becomes[1]

$$u_t + 6uu_x + u_{xxx} = 0 \tag{2}$$

Soliton solution begins when Zabusky and kruskal (1965) led to work on KdV equation[2]. They studied the initial value problem to KdV equation[4] and found that the wave steepened and produced a shock. The dispersive term, δu_{xxx} then became significant and balanced the nonlinearity. After a few times, the solutions developed a wave of eight well-defined, each like *sech*^{2} function where the faster (taller) waves chased and overtook the slower (shorter) waves. These waves interacted and merged with each other without change of form and amplitude, but with a small change of their phase. Because of this observation, Zabusky and Kruskal named these waves as *solitons*[3].

In this work, equation (2) has been identified to be solved analytically. One-soliton solution has been produced by using separation of variable method meanwhile Backlund Transformation and Nonlinear Superposition Principle produce two-soliton and three-soliton solutions. In order to study the interaction of these solitary wave, two-soliton and three-soliton have been plotted graphically using *Mathematica*.

2. Solutions of KdV Equation

In order to solve equation (2), we begin with a trial solution of the form $u(x,t) = z(\xi)$ where $\xi = x - 2\beta t$ which corresponds to a constant wave form moving with velocity, 2β. Substituting $u(x,t) = z(x - 2\beta t)$ and $\xi = x - 2\beta t$ into every term of equation (2) gives

$$-2\beta z' + 6zz' + z''' = 0\,. \tag{3}$$

Integrating equation (3) with respect to ξ gives

$$-2\beta z + 3z^2 + z'' = A\,. \tag{4}$$

where A is a constant of integration. Multiplying equation (4) by z' and integrate it respect to ξ to obtain

$$-\beta z^2 + z^3 + \frac{1}{2}(z')^2 = Az + B. \tag{5}$$

where B in a second constant of integration. We now require that z, $\frac{dz}{d\xi}$ and $\frac{d^2z}{d\xi^2}$ to vanish as $x \to \pm\infty$. From equation (4) and equation (5), A = 0 and B = 0. Thus, equation (5) becomes a separable equation of first-order ordinary differential equation. It can be written as,

$$\frac{dz}{d\xi} = z\sqrt{2\beta - 2z} \tag{6}$$

Separating equation (6) become

$$\int \frac{dz}{z\sqrt{2\beta - 2z}} = \int d\xi \tag{7}$$

To solve the integration of the left-hand side of equation (7), we use the hyperbolic trigonometry substitution[5], $z = \beta \operatorname{sech}^2 w$. Substituting this function in every term of equation (7) and solve it, we get

$$-\frac{2}{\sqrt{2\beta}}(w) = \xi + a \tag{8}$$

a is a constant of integration where it assumed to be 0.

In order to obtain solution in function x and t, substitute back trial solution and trigonometry substitution into equation (8). Finally we get

$$u(x,t) = \beta \operatorname{sech}^2\left[\sqrt{\frac{\beta}{2}}\left(x - 2\beta t\right)\right] \tag{9}$$

where equation (9) is called one-soliton solution of KdV equation. We have constructed two-soliton and three-soliton solution by using Backlund Transformation and Nonlinear Superposition Principles[5]. Hence, we get two-soliton solution for KdV equation as

$$u(x,t) = -2(\beta_1 - \beta_2)\frac{\beta_1 A^2(x,t) + \beta_2 B^2(x,t)}{\left[\sqrt{2\beta_1}C^2(x,t) - \sqrt{2\beta_2}D^2(x,t)\right]^2} \tag{10}$$

where

$$A(x,t) = \sec h\left[\sqrt{\beta_1/2}\left(x - 2\beta_1 t\right)\right]$$

$$B(x,t) = \operatorname{csch}\left[\sqrt{\beta_2/2}\left(x - 2\beta_2 t\right)\right]$$

$$C(x,t) = \tanh\left[\sqrt{\beta_1/2}\left(x - 2\beta_1 t\right)\right]$$

$$D(x,t) = \coth\left[\sqrt{\beta_2/2}\left(x-2\beta_2 t\right)\right]$$

Meanwhile, three-soliton solution for KdV equation as

$$u(x,t) = \beta_1 A^2(x,t) - 2(\beta_2-\beta_3)\left[\frac{2(\beta_3-\beta_1)P - 2(\beta_1-\beta_2)Q}{\left(R-S\right)^2}\right] \tag{11}$$

where

$$P = \frac{\beta_3 \operatorname{sec}h^2\left[\sqrt{\beta_3/2}\left(x-2\beta_3 t\right)\right] - \beta_1 \operatorname{sec}h^2\left[\sqrt{\beta_1/2}\left(x-2\beta_1 t\right)\right]}{\left[\sqrt{2\beta_3}\tanh\left[\sqrt{\beta_3/2}\left(x-2\beta_3 t\right)\right] - \sqrt{2\beta_1}\tanh\left[\sqrt{\beta_1/2}\left(x-2\beta_1 t\right)\right]\right]^2}$$

$$Q = \frac{\beta_2 \operatorname{csch}^2\left[\sqrt{\beta_2/2}\left(x-2\beta_2 t\right)\right] - \beta_1 \operatorname{sech}^2\left[\sqrt{\beta_1/2}\left(x-2\beta_1 t\right)\right]}{\left[\sqrt{2\beta_1}\tanh\left[\sqrt{\beta_1/2}\left(x-2\beta_1 t\right)\right] - \sqrt{2\beta_2}\coth\left[\sqrt{\beta_2/2}\left(x-2\beta_2 t\right)\right]\right]^2}$$

$$R = \frac{2(\beta_1-\beta_2)}{\sqrt{2\beta_1}\tanh\left[\sqrt{\beta_1/2}\left(x-2\beta_1 t\right)\right] - \sqrt{2\beta_2}\coth\left[\sqrt{\beta_2/2}\left(x-2\beta_2 t\right)\right]}$$

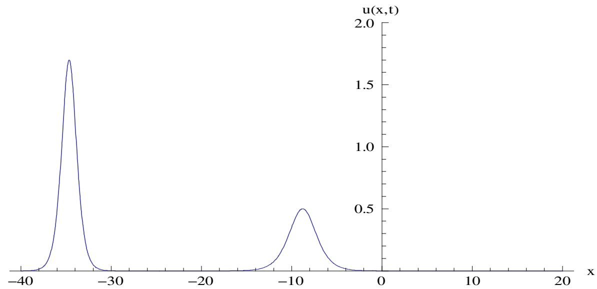
$$S = \frac{2(\beta_3-\beta_1)}{\sqrt{2\beta_3}\tanh\left[\sqrt{\beta_3/2}\left(x-2\beta_3 t\right)\right] - \sqrt{2\beta_1}\tanh\left[\sqrt{\beta_1/2}\left(x-2\beta_1 t\right)\right]}$$

Two-soliton means two solitary waves moving at same time with different speed, $2\beta_1$ and $2\beta_2$ where $2\beta_2$ move faster than $2\beta_1$ because $\beta_2 > \beta_1$. Meanwhile, three-soliton solution means that three solitary waves are moving at the same time, each with different speed, $2\beta_1, 2\beta_2$ and $2\beta_3$ where $2\beta_3 > 2\beta_2 > 2\beta_1$.

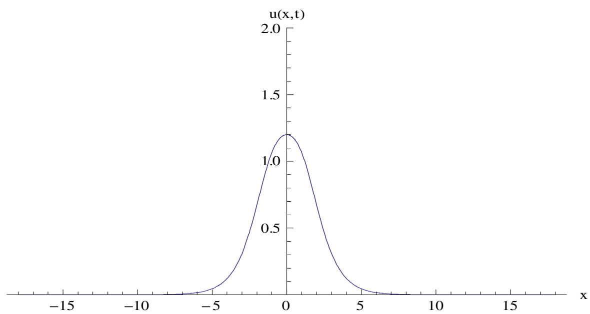
3. Interaction of Two-Soliton and Three-Soliton

Solutions of KdV equation have been obtained as equation (9), equation (10) and equation (11). In order to study the soliton interaction, two-soliton and three-soliton solutions at different value of *t* have been plotted.

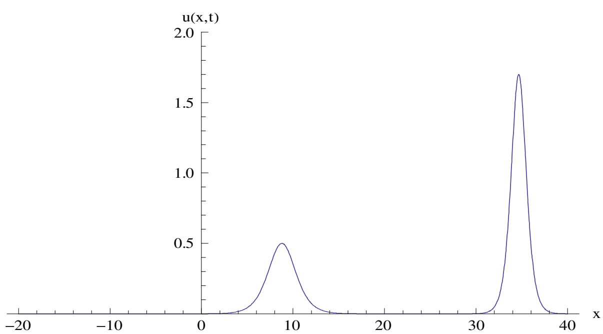
Figure 1 shows two solitary wave propagate along x-axis with amplitude equal to 0.5 and 1.7, respectively. Initially, these two waves propagate apart as shown in Fig. 1(a) where the taller wave is behind the shorter wave. Figure 1(b) shows that two solitary waves in Fig. 1(a) are in interaction condition. As time increases, the taller wave moves forward and leaves the shorter wave behind. This can be seen in Fig. 1 (c).



(a) $t = -10$



(b) $t = 0$



(c) $t = 10$

Fig. 1. Propagation of two-soliton solution when $\beta_1 = 0.5$ and $\beta_2 = 1.7$.

Figure 2 illustrates propagation of the taller wave with amplitude 3, the medium wave with amplitude 2.5 and the shorter wave with amplitude 0.5. When $t = -60$, the shorter wave lead the medium wave and the medium wave lead the taller wave. It can be seen in Fig. 2(a). Figure 2(b) shows that these three solitary waves merge to each other. As time increases, these waves begin to move separately. Figure 2(c) shows the taller wave and the medium wave continue to move toward far away from the shorter wave.

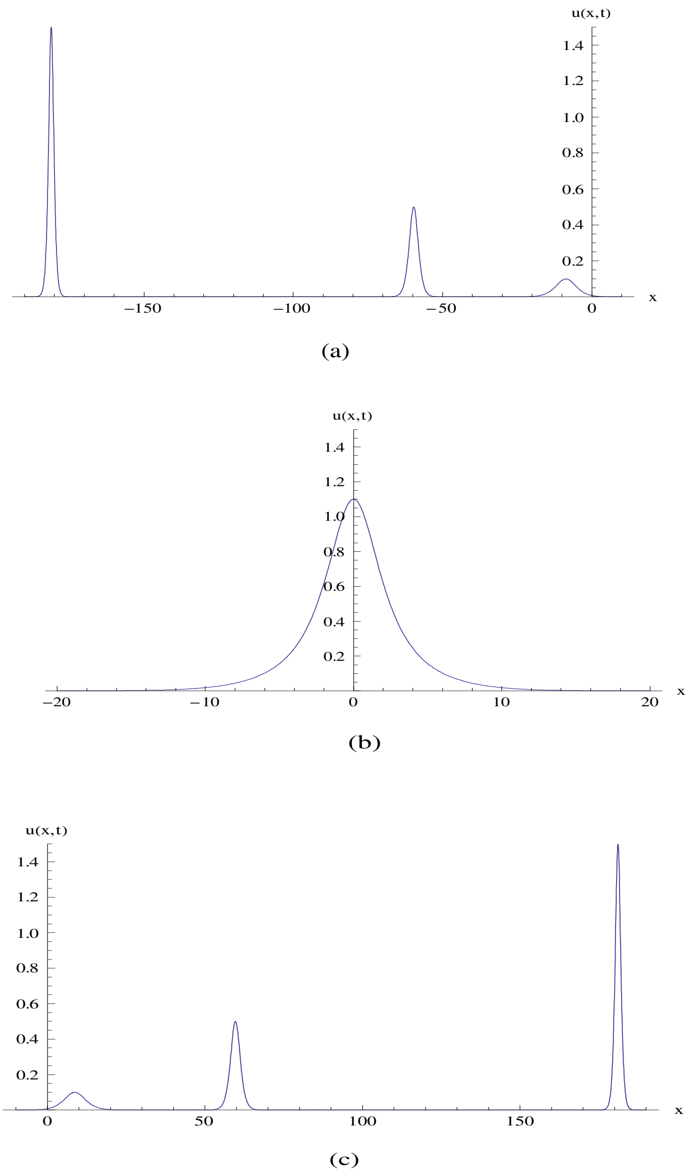


Fig. 2. Propagation of three-soliton solution when $\beta_1 = 0.5$, $\beta_2 = 2.5$ and $\beta_3 = 3$.

4. Conclusion

In this work, one-soliton solution for KdV equation has been solved using Separation of Variable method. Then this solution is extended to two- and three-soliton solution. Previously defined that the speed of solitary waves is defined as 2β. Hence, we can say that the speed is proportional to the amplitude where the wave with higher amplitude moves faster than the wave with lower amplitude. In order to study the waves behavior before, during and after interaction, we chose $t < 0$ as before the interaction, $t = 0$ as during interaction and $t > 0$ as after the interaction.

It has been noticed that before the interaction, the wave with higher amplitude is behind the wave with lower amplitude. As time increases, the wave with higher amplitude moves faster and chases the wave with lower amplitude until they collided. After they have interacted with each other, the wave with higher amplitude overtook the wave with the lower amplitude and leaves the slower wave behind. This phenomenon is shown through the simulation in Fig. 1 and Fig. 2. The interactions also showed that these waves preserved its physical properties such as amplitude and speed before and after interaction. This is the main characteristic of soliton. Hence, we concluded that solutions to KdV equation obtained in this work are soliton solutions.

Acknowledgments

The authors would like to thank Univeristi Malaysia Perlis for the financial support of this work through Short Term Grant 2010/2011.

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