Integrating Mathematics and Physics in the Learning of Oscillations

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1. Introduction

Mathematics and Physics

Knowledge in mathematics is a prerequisite in the learning of science. In Physics, learning is based on the detailed analysis of mathematical solutions to specific problems (Sharlik and Dejong, 1986). The integrated mathematics and physics lesson approach is illustrated in several researches (John and Rebecca, 1995) and additional benefits for the course were proposed (Fathe et.al, 1998; Westbrook, 1998; Hanson, 1998). For example, students who had studied arithmetic progressions were able to recognize a physics problem immediately and found the solutions by straightforward application of the algebraic methods (Bassok and Holyoak, 1989). In contrast, those who do not really understand sine and cosines as oscillatory functions face obstacles in studying oscillations and waves (Knight, 2002).

Learning Mathematics and Physics in OUM

In Open University Malaysia (OUM), the science stream students are required to take a basic mathematics subject before taking physics. However some of them have no confidence to learn mathematics and worst still some even hate mathematics. They are not aware of the use of mathematics in the learning of physics. This might be due to the differences in the language, notation and approach used in the two disciplines. In addition, these two subjects are learned separately at different time by these distance learners who practice self and independent learning based on limited sources. Thus, it is hard to convince them that there is an interaction between mathematics and physics. Students frequently find difficulties to effectively integrate the mathematics with science and to apply it in their subsequent coursework (Hanson, 1998).

Research Purpose

This study is carried out to investigate the integration of mathematics and physics, which only concentrated on the topic of oscillation and covered displacement, velocity, acceleration and energy in simple harmonic motion (SHM). The objectives of the study are 1) to identify the application of mathematical concepts and skills in oscillation and 2) to reveal the mathematical difficulties in SHM problem solving 3) to interpret the role of mathematics in the learning of oscillation. This study is considered valid to distance learners at OUM who may have overlooked the importance of mathematics to physics. The finding is expected to change the students' perception on mathematics. This is necessary because an improvement in mathematics will be one of the ways to improve their performance in physics.

Research Method

This study focused on the science stream students in OUM. These students had passed Basic Mathematics subject at the first academic semester and taken the physics subject at second semester. Since this study only looks into the topic of oscillation, only the subtopics of displacement, velocity, acceleration and energy in simple harmonic motion (SHM) are investigated. The content of the printed module used by the students was observed. Then, a related problem was selected to investigate students' solutions and to detect their mistakes.

2. Utilization of Mathematics in the Learning of Oscillation

The oscillations of a block attached to a spring, the swinging pendulum of a clock and the vibrations produced by a stringed musical instrument are examples of periodic motion familiar to most students. In physics, simple harmonic motion (SHM) is a very special kind of motion. An effective mathematical model can be utilized because it can state the variable properties associated with SHM in a simple form which can be applied in studying other oscillating systems as well.

A particle undergoing SHM will tend to oscillate back and forth about a fixed point indefinitely in the absence of friction. A good example of this is the oscillation of a block of mass m attached to a horizontal spring with a force-constant k i.e. the spring-block system. Once the block is displaced by a small distance x from its equilibrium position and released, it will oscillate back and forth about the equilibrium position *0*. The basic characteristics for SHM are as follows:

(i) the acceleration of the block $\frac{d^2x}{dt^2}$ is proportional to the variable x

ii) $\frac{d^2x}{dt^2}$ is always directed toward *0*. Put another way, $\frac{d^2x}{dt^2}$ and x are in *opposite* directions.

Mathematically, these characteristics can be written in a compact form that involves a second order differential equation that describes the equation of motion for SHM:

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad \text{.................................Eq.1}$$

where $\omega^2 = \frac{k}{m}$ is a constant. The negative sign indicates opposite directions as stated earlier.

The solution to Eq.1 should involve a mathematical function whose condition is such that its second derivative is equal to the original function with a negative sign multiplied by ω^2.

The trigonometric functions sine and cosine are unique in the sense that they fulfill this special condition, so the solution to Eq.1 can be constructed around one or both of these. By substitution, students can easily verify that the following are general solutions to Eq.1:

$$x = A\cos(\omega t + \phi) \quad \text{...........................Eq.2}$$

or

$$x = A\sin(\omega t + \phi) \quad \text{.............................Eq.3}$$

In Equations 2 and 3, A is the maximum displacement (or the amplitude) of the block from the equilibrium position, ω is the angular frequency of the block, and ϕ is the phase angle in radians. A, ω and ϕ are all constants. Either of these equations may be used to represent a body in SHM.

The mathematical expressions for the remaining two variables in SHM, the velocity v and acceleration a, as functions of time t can then be obtained from Eq.2 or Eq.3. Besides this, students also prefer to graph these three variables as functions of time. The advantage of this approach is that it helps the student develop a better mental picture of SHM.

Another application of mathematics is found in the calculation of the total mechanical energy of the spring-block system. The total mechanical energy E is obtained by adding the kinetic energy K of the oscillating block with the potential energy of the spring U. By using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, it can be shown that the total mechanical energy in SHM is constant; this is consistent with the Conservation of Energy Principle.

3. Mathematical Difficulties Encountered in Solving a Problem in SHM

A block of mass m is undergoing SHM with amplitude A on the end of a horizontal spring. The displacement of the block is given by

$$x = A\cos(\omega t + \phi)$$

where x is in cm and t in seconds. You may set $\phi = 0$ when you sketch the graphs.

a) Show on a graph how the displacement varies with time.
b) What should the values of ϕ be to transfer x into a sine function.
c) Find expressions for the velocity, v and acceleration, a of the block.
d) Sketch the graphs for the velocity, v and the acceleration, a versus time.
e) Find the maximum velocity.
f) Show that the total mechanical energy of the spring-block system is constant.

Mathematical notation and physical situation

Sinusoidal functions i.e. sine and cosine functions are widely used in the study of oscillations but some students may not know enough about these functions. They think that it is sufficient merely to confirm that a sinusoidal function satisfies the equation of motion. They may not be able to interpret the physical significance from the mathematical equation, $x = A\cos(\omega t)$. In fact, ω is determined by the physical characteristics of the oscillator k and m for a spring, L and g for a pendulum. The amplitude A and phase constant ωt can have any values. The significance of this fact will help them to sketch the graph. In contrast, they may determine the incorrect variable for x-axis and y-axis of the graph and sketched an incorrect graph.

Properties of trigonometry functions

In order to simplify the process of sketching a graph, the general shape of the graph and the properties of particular function should be known before hand. In this case, the general graph and properties of the trigonometric functions are required. To sketch the graph of the displacement versus time, the graphs of $x = \cos t$ (Figure 1) was referred, where the maximum value is 1. Thus, a similar graph can be sketched but with a maximum value of A to obtain the required graph of $x = A\cos(\omega t)$ (Figure 2).

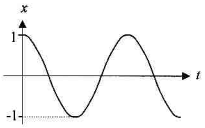

Figure 1: $x = \cos t$

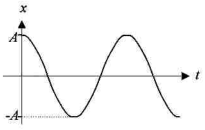

Figure 2: $x = A\cos(\omega t)$

However, some students were unable to recall the properties for the cosine function from their prior knowledge of mathematics. Others sketched incorrect graphs because they mixed up the sine and cosine functions. Few even thought that the two functions are equal! Some did not understand the properties of the function and showed the incorrect maximum and minimum values of the graph. Other students viewed the algebraic trigonometry function as a syntactic notation. Actually, there is a hidden meaning behind the sine function which can help determine the maximum velocity. The maximum value for $\sin(\omega t)$ is 1, thus the maximum value for the velocity, $v = -A\omega\sin(\omega t)$ is $-A\omega(1) = -A\omega$.

Graphical representation and its physical behaviour

Alternatively, students can find out the maximum velocity by comparing the sketched graphs of displacement and velocity. At $x = 0$ i.e. when the block is passing through the equilibrium position, the velocity is maximum. It is also seen that when the velocity is maximum, the acceleration is minimum and vice-versa (Figure 3). However, this can only be noticed when it is displayed graphically. Unfortunately, students, who cannot see the relationship between the mathematics and physics, will not be able to make these conclusions.

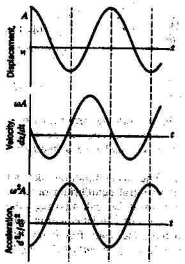

Figure 3

Graph transformation and algebraic expression

From the derivative rules on the trigonometry functions, $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$, students may see the link between the sine and cosine functions. However, some of the students are unable to relate the graphical representation of the sine function to the graph of the cosine function through graph transformation. This is needed when expressing the displacement, $x = A\cos(\omega t)$ as a sine function through graphical representation. Geometrically, first shift the graph of the displacement by $90°$ to the right to obtain $x = A\cos\left(\omega t - \frac{\pi}{2}\right)$. By expanding this expression using the trigonometric identity $\cos(A+B) = \cos A\cos B - \sin A\sin B$ one obtains $x = A\sin(\omega t)$. Therefore the value of ϕ is indeed $\frac{\pi}{2}$ rad. Clearly, sine and cosine are the same oscillatory function with different phase constants. Students who thought that sine and cosine were two distinct functions could not recognize this.

Derivative

Students who are weak in differentiation faced difficulties in finding the correct expressions for the velocity and acceleration. Actually, the expression of velocity can be determined quite easily by differentiating the displacement with respect to *t*. In a similar manner, the expression for the acceleration can be obtained from the velocity. For those who only memorize the differentiation of sine and cosine function as $\frac{d}{dt}(\sin t) = \cos t$ and $\frac{d}{dt}(\cos t) = -\sin t$ without truly understanding the properties of the derivative rules on the trigonometric functions will not be able to get the right solutions.

Symbol manipulation in algebraic expressions

Without a strong foundation in algebra, students may easily get bogged down by the complicated symbol manipulation when expanding the algebraic expressions $(-A\omega\sin(\omega t+\phi))^2$ of the kinetic energy, *K* and the potential energy, *U*. If the students only memorize the trigonometric identity, $\sin^2\theta + \cos^2\theta = 1$, without applying the exponential interpretation, $\sin^2\theta = \sin\theta\cdot\sin\theta$. they would not able to prove that the total mechanical energy in SHM,

$\frac{1}{2}kA^2\left(sin^2(\omega t+\phi)\right) + \frac{1}{2}kA^2\left(cos^2(\omega t+\phi)\right)$ is constant.

4. Role of mathematics in Oscillations

From the mathematical application and the difficulties identified in the problem related to SHM, the role of mathematics can now be divided into three categories:

4.1 Language

Science is about discerning and representing structure, while mathematics has been called the 'language of structure' (MacDuff, 2003). This is fulfilled when a general mathematical language is used to describe several models of SHM although the specific details depend on the situation. The mathematical description that was developed for the spring-block system can be used as a model to describe the behavior of other physical systems like the simple pendulum and a mass hanging from a vertical spring.

Basically, all of the motion in SHM can be generalized by a sinusoidal function, $x = A\cos(\omega t + \phi)$. The connection between A, ω and ϕ is expressing through this function.

4.2 Alternative Approach

The graphical representation is another approach to understand well the displacement, velocity and acceleration of a body in simple harmonic motion besides the algebraic representation. Both of the approaches are important because the harmonic function defined by algebraic expression represented symbolically is easier to manipulate and analyze. Meanwhile, visualizing a function by graph makes the algebraic symbol significant (Buck, 2000). Thus, the students are able to 'see' the motion through the graph of the displacement versus time. Moreover, it was shown graphically in Figure 3 that the block achieved maximum velocity whenever it passed through the equilibrium position $x = 0$. As shown in the results, the concept, theory or the law of physics become very clear when graphs are used. Obviously incorrect graphs and transformation will lead to incorrect physical interpretation.

4.3 Technique

Mathematical techniques that are developed must be able to describe physical theories and their consequences. For example, first derivative is a technique that led to the expression of velocity based on the expression of displacement. The second derivative can also be applied to the displacement to get the acceleration. Both of the techniques can be used to get the "target" i.e. acceleration. Thus, the suitable technique should be chosen.

5. Conclusion

Generally, algebra, geometry, trigonometry and calculus are the obvious mathematical applications in oscillations. The utilization of mathematics in oscillation helps the students to sketch and interpret oscillatory graphs, to obtain the expressions of velocity and acceleration and to show the total mechanical energy in SHM is constant. On the other hand, because of the mathematical difficulties in terms of mathematical notation, symbol manipulation, algebraic expression, graph transformation, trigonometry function and derivative, students quite often make mistakes and develop misconceptions about oscillations. Actually, mathematics can be described as a language, an alternative approach and a technique; each having a specific role to play in the learning of oscillation. In this way, physical situation of oscillation can be expressed clearly in a simple form and multiple representations. The physical ideas can also be expanded after the adequate mathematical analysis.

It is suggested from this paper that an integrated mathematics and physics course be introduced for learning oscillations. Its effectiveness can be verified when the performance of students in the learning of oscillations is satisfactory. Thus, it may be expanded to other topics of physics.

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